

PAR–LPTHE 02–06
hep-th/0207020

Eight-Dimensional Topological Gravity and its Correspondence with Supergravity

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Abstract: A topological theory for euclidean gravity in eight dimensions is built by enforcing octonionic self-duality conditions on the spin connection. The eight-dimensional manifold must be of a special type, with $G_2 \subset Spin(7) \subset SO(8)$ holonomy. The resulting theory is related to a twisted version of $N = 1$, $D = 8$ supergravity. The situation is comparable to that of the topological Yang–Mills theory in eight dimensions, for which the $SO(8)$ invariance is broken down to $Spin(7)$, but is recovered after untwisting the topological theory.

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1 Introduction

We have shown in [1] that topological gravity in four dimensions can be identified with the $N = 2$, $D = 4$ supergravity, for manifolds with special holonomy. Our inspiration was that superstring theories can be obtained, at least in a formal way, as suitable anomaly free untwisting of topological sigma-models [2], so that supergravities, which arise as low energy limits of superstrings, should be linked to topological gravities.

Here we consider the case of eight-dimensional gravity, with the ultimate goal of showing that $D = 11$ supergravity, which determines all known supergravities in lower dimensions, can be viewed as a topological theory. We indicate in this note some very encouraging results in this direction. We show how the Einstein action plus the Rarita–Schwinger term can be obtained in eight dimensions by imposing in a BRST invariant way the octonionic gravitational instanton equation, generalizing the $N = 2$, $D = 4$ supergravity case worked out in [1, 3]. As was shown in [4], going from four dimensions to eight dimensions amounts to changing the quaternionic structure coefficients into octonionic ones in the self-duality equations that are used to express the topological field theory. From the beginning, this imposes that one uses a manifold with special holonomy group, $H \subset SO(8)$. Actually, we will reach the conclusion that the topological field theory must be considered on a manifold with $G_2 \subset Spin(7) \subset SO(8)$ holonomy.

As we will discuss in sect. 4, the topological model that we consider is related to a twisted version of $N = 1$, $D = 8$ supergravity. It thus can be viewed as an effective supergravity of heterotic string theory. This is in line with the results of [5], where the twist of the four dimensional effective theories of heterotic superstrings has been discussed. We leave to another paper a delicate point, the fact that the presence of unpaired fields in the topological model should require the introduction of a new sector involving a two-form gauge field, completing the identification with supergravity. This question is under consideration but its technical complications deserve a separate publication.

2 The octonionic self duality equation and the Einstein action in eight dimensions

As in four dimensions, we would like to build a topological field theory by enforcing self-duality conditions on the spin connection ω , taken as a functional of the vielbein e [1]. In eight dimensions, a natural choice is the octonionic gravitational instanton equation [6], given by

$$\omega^{ab} - \frac{1}{2}\Omega^{abcd}\omega^{cd} = 0, \quad (1)$$

where Ω is a completely antisymmetric four-tensor invariant under a $Spin(7)$ subgroup of $SO(8)$. Indeed, the four-tensor Ω induces a $Spin(7)$ invariant decomposition of the adjoint representation of $SO(8)$, $\mathbf{28} = \mathbf{7} \oplus \mathbf{21}$ [4]. Eq. (1) corresponds to setting to zero the components ω^{ab^-} of the spin connection in the seven dimensional subspace; it then counts for $56 = 7 \times 8$ conditions, which exactly match the number of degrees of freedom contained in the vielbein e_μ^a modulo reparametrizations.

A remarkable fact is that, as happens in the four-dimensional case [1], the eight-dimensional Einstein Lagrangian can be expressed, up to a pure derivative, as a quadratic form in the anti-self-dual part of the spin connection:

$$\mathcal{L}_{EH} = \frac{1}{2}R^{ab}\mathcal{V}_{ab} = 2\omega^{ac^-}\omega^{bc^-}\mathcal{V}_{ab} + 2d(\omega^{ab^-}\mathcal{V}_{ab}), \quad (2)$$

where R is the Lorentz curvature and

$$\mathcal{V}_{ab} = \frac{1}{6!}\epsilon_{abcdefghj}e^ce^de^fe^ge^he^j. \quad (3)$$

Notice that each of the two terms in the right hand side of eq. (2) explicitly displays only $Spin(7)$ gauge invariance. Thus, this expression of the Einstein action is globally well defined only on manifolds of $Spin(7)$ holonomy. From eq. (2) follows that any solution of eq. (1) is a stationary point of the Einstein action and a solution of the Einstein equations. This suggests that the topological field theory that we shall build using eq. (1) as gauge fixing contains ordinary gravitation.

The proof of eq. (2) starts from the Bianchi identity, $DT^a = R^{ab}e^b = 0$, which implies $\Omega R^{ab}e^be^a = 0$. This can be rewritten as

$$\Omega_{abcd}R^{cd}\mathcal{V}^{ab} = 0, \quad (4)$$

using the fact that Ω is a self dual 4-form. We then decompose R^{ab} under $Spin(7)$, writing

$$R^{ab} = R^{ab+} + R^{ab-}, \quad \frac{1}{2}\Omega_{abcd}R^{ab} = R^{cd+} - 3R^{cd-}. \quad (5)$$

Eq. (5) and the Bianchi identity expressed in eq. (4) allows for the elimination of R^{ab+} in the Einstein Lagrangian:

$$\frac{1}{2}R^{ab}\mathcal{V}_{ab} = 2R^{ab-}\mathcal{V}_{ab} \quad (6)$$

Using that $R^{ab-} = d\omega^{ab-} + \frac{1}{2}[\omega, \omega^-]^{ab} - \omega^{ac-}\omega^{cb-}$ and the zero-torsion condition, one finally gets the very interesting identity of eq. (2).

3 The fields of topological gravity

We now proceed to the construction of the topological field theory based on the gauge-fixing condition (1). The set of fields of the TQFT is the eight-dimensional generalization of that used in [1]:

$$\begin{array}{ccccc} & & e_\mu^a & & \\ & \Psi_\mu^a & & \bar{\Psi}_\mu^{ab-} & \\ \Phi^a & & L^{ab-}, b_\mu^{ab-} & & \bar{\Phi}^a \\ & \eta^{ab-} & & \bar{\eta}^a & \end{array} \quad (7)$$

$$\begin{array}{ccccc} & & \omega_\mu^{ab} & & \\ & \tilde{\Psi}_\mu^{ab} & & \bar{\tilde{\Psi}}_\mu^{ab} & \\ \tilde{\Phi}^{ab} & & \tilde{L}^{ab}, \tilde{b}_\mu^{ab} & & \bar{\tilde{\Phi}}^{ab} \\ & \tilde{\eta}^{ab} & & \bar{\tilde{\eta}}^{ab} & \end{array} \quad (8)$$

$$\begin{array}{ccccc} & & A_\mu & & \\ & \bar{\Psi}_\mu & & \chi^{ab-} & \\ \Phi & & \beta^{ab-} & & \bar{\Phi} \\ & & \eta & & \end{array} \quad (9)$$

$$\begin{array}{ccccccc} \xi^\mu & & \bar{\xi}^\mu & & \Omega^{ab} & & \bar{\Omega}^{ab} \\ & b^\mu & & & b^{ab} & & c \\ & & & & & & b \quad \bar{c} \end{array} \quad (10)$$

In the above tables, ghost numbers increase to the left and the BRST transformation acts in the south-west direction.

The Lorentz×diffeomorphism symmetry determines a Lie group and its associated topological BRST symmetry can be constructed as:

$$\begin{aligned}
se_\mu^a &= \mathcal{L}_\xi e_\mu^a - \Omega^{ab} e_\mu^b + \Psi_\mu^a \\
s\omega_\mu^{ab} &= \mathcal{L}_\xi \omega_\mu^{ab} + D_\mu \Omega^{ab} + \tilde{\Psi}_\mu^{ab} \\
s\Psi_\mu^a &= \mathcal{L}_\xi \Psi_\mu^a - \Omega^{ab} \Psi_\mu^b - \mathcal{L}_\Phi e_\mu^a + \tilde{\Phi}^{ab} e_\mu^b \\
s\tilde{\Psi}_\mu^{ab} &= \mathcal{L}_\xi \tilde{\Psi}_\mu^{ab} - [\Omega, \tilde{\Psi}_\mu]^{ab} + D_\mu \tilde{\Phi}^{ab} - \mathcal{L}_\Phi \omega_\mu^{ab} \\
s\Phi^a &= \mathcal{L}_\xi \Phi^a - \Omega^{ac} \Phi^c \\
s\tilde{\Phi}^{ab} &= \mathcal{L}_\xi \tilde{\Phi}^{ab} - [\Omega, \tilde{\Phi}]^{ab} \\
s\xi^\mu &= \Phi^\mu + \xi^\nu \partial_\nu \xi^\mu \\
s\Omega^{ab} &= \mathcal{L}_\xi \Omega^{ab} - \Omega^{ac} \Omega^{cb} + \tilde{\Phi}^{ab} .
\end{aligned} \tag{11}$$

Moreover, we have that $\Phi^a = e_\mu^a \Phi^\mu$, and thus, $s\Phi^\mu = \mathcal{L}_\xi \Phi^\mu$. The other fields in eq. (7), (8) and (10) are doublets of antighosts and Lagrange multipliers. Their BRST transformations are given by

$$\hat{s}\bar{g} = \lambda \quad \hat{s}\lambda = \mathcal{L}_\Phi \bar{g} + \delta_{\tilde{\Phi}} \bar{g}[\tilde{\Phi}, \bar{g}] , \tag{12}$$

with $sX = \hat{s}X + \mathcal{L}_\xi X + \delta_\Omega X$ and δ_Ω a Lorentz transformation with parameter Ω .

Finally, the BRST transformations for the fields of the $U(1)$ gauge sector are

$$\begin{aligned}
sA_\mu &= \mathcal{L}_\xi A_\mu + \partial_\mu c + \bar{\Psi}_\mu \\
s\bar{\Psi}_\mu &= \mathcal{L}_\xi \bar{\Psi}_\mu - \partial_\mu \Phi - \mathcal{L}_\Phi A_\mu \\
s\Phi &= \mathcal{L}_\xi \Phi - \mathcal{L}_\Phi c \\
sc &= \mathcal{L}_\xi c + \Phi .
\end{aligned} \tag{13}$$

The fields in eqs. (7–10) have a suitable decomposition under $Spin(7)$, according to the choice of the gauge-fixing condition (1).

The fields $(\bar{\Psi}_\mu^{ab-}, b_\mu^{ab-})$ in (7) are respectively the antighost and Lagrange multiplier associated to the gauge-fixing condition (1), and thus belong to the seven dimensional anti-self-dual representation of $Spin(7)$. Moreover, the presence of the $U(1)$ topological multiplet implies a further decomposition

of the twenty-one-dimensional self-dual representation of $Spin(7)$ under a G_2 subgroup, $\mathbf{21} = \mathbf{7} \oplus \mathbf{14}$. The fields χ^{ab+} and β^{ab+} in (9) belong to the seven-dimensional subspace of this G_2 decomposition.

In the following section we will describe the topological theory that can be built from the fields in eqs. (7–10) and its relationship with a suitable twisted version of $N = 1$, $D = 8$ supergravity.

4 Topological action and its relationship with twisted supergravity

In order to implement eq. (2), we consider the following action, for a manifold with $Spin(7)$ holonomy:

$$I = \int_{\mathcal{M}_8} s \left(\bar{\Psi}^{ab-} (\omega^{ac-} + b^{ac-}) \mathcal{V}_{bc} + F \Psi^a e^a \right) \quad (14)$$

The action eq. (14) defines a cohomological theory, whose BRST symmetry is associated to a $Spin(7) \times \text{diffeomorphism} \times \text{gauge invariance}$. By expanding the first term of eq. (14) one gets the bosonic action

$$I_1 = \int_{\mathcal{M}_8} b^{ab-} b^{ac-} \mathcal{V}_{bc} + b^{ab-} \omega^{ac-} \mathcal{V}_{bc} \ , \quad (15)$$

which reproduces the Einstein action after elimination of the field b , according to the identity (2). A natural question now arises, whether the topological model we are considering can be identified with a suitable twisted version of eight-dimensional supergravity. This turns out to be actually the case. Notice in fact that on a manifold with $Spin(7)$ holonomy there exist a covariantly constant spinor (of norm one) η , which can be used to redefine the gravitino $\bar{\Lambda} = (\lambda, \bar{\lambda})$ of $N = 1$, $D = 8$ supergravity as

$$\begin{aligned} \lambda &= \Psi^a \gamma_a \eta \ , \\ \bar{\lambda} &= \bar{\Psi} \eta + \bar{\Psi}^{ab-} \gamma_{ab} \eta \ , \end{aligned} \quad (16)$$

where $(\lambda, \bar{\lambda})$ are Weyl spinors of opposite chiralities and the eight-dimensional gamma matrices γ_a acts on spinors of definite chirality. Moreover, we have [4]

$$\Omega_{abcd} = \eta^T \gamma_{abcd} \eta. \quad (17)$$

From eq. (16) we see that the gravitino is then mapped to the fields $(\Psi^a, \bar{\Psi}, \bar{\Psi}^{ab-})$ of the topological model. This suggest to compare the topological action (14) with a twisted version of $N = 1, D = 8$ supergravity. In the spirit of topological field theory, we can restrict our attention on the kinetic terms for the fields, which simplifies considerably the comparison. In fact, the basic requirement on the gauge-fixing conditions is that they must give a good definition for the propagators of the fields. Interaction terms can be then always added as BRST exact terms in order to get agreement with the complete twisted supergravity action. The kinetic terms for the fields $(\Psi^a, \bar{\Psi}, \bar{\Psi}^{ab-})$ in the topological action eq. (14) are given by

$$I_2 = \int_{\mathcal{M}_8} \left[\sqrt{g} d^8x \left(\bar{\Psi}_a^{ac-} (\partial_{[b} \Psi_{c]}^b + \frac{1}{2} \Omega_{cbfg} \partial_b \Psi_f^g) - \frac{1}{2} \Psi_b^{ac-} \partial_{[a} \Psi_{c]}^b \right. \right. \\ \left. \left. - \frac{1}{2} \bar{\Psi}_b^{ac-} \Omega_{cbfg} (\partial_{[a} \Psi_{f]}^g - \partial_f \Psi_g^a) \right) + \Omega \bar{\Psi} d\Psi^a e^a \right] \quad (18)$$

After some algebra on the gamma matrices and using eq. (17) and the self-duality of the four-form Ω , one can verify that the action eq. (18) corresponds to the quadratic part of the ordinary Rarita-Schwinger action

$$I_{RS} = \int \sqrt{g} d^8x \bar{\Lambda}_a \Gamma^{abc} \partial_b \Lambda_c, \quad (19)$$

written in terms of the twisted variables eq. (16).

Finally, we add to the action eq. (14) the term

$$I_{gauge} = \int_{\mathcal{M}_8} s \left[\Omega \chi^+ (\beta^+ + F) \right] \quad (20)$$

which, upon integration on the auxiliary field β^+ , gives the right kinetic term for the graviphoton field A [4]. Notice that the presence in eq. (20) of the fields (χ^+, β^+) breaks the $Spin(7)$ symmetry down to G_2 .

Summarizing, we have seen that the topological model defined by the action eq. (14) and eq. (20) corresponds to the twisted version of a gravitational model containing a graviton, a gravitino and a graviphoton. In order to make a full comparison with $N = 1, D = 8$ supergravity, we should still introduce in the topological field theory a sector describing a two-form field and the corresponding gauge field entering the definition of the curvature three-form [7]. We deserve the analysis of this sector for further work.

Here we only remark that the gauge-fixing of the $Spin(7)$ symmetry of our model already gives indications on the presence of a further topological

multiplet in order to have a well defined functional integral for all the fields introduced in sect.3. In fact, in defining the topological action, one must use all fields that occur in the topological quartets in eq. (7-10). Let us describe in detail how this is done.

First of all, the topological freedom in ω is used to constrain the torsion $T = de + \omega e$ and determine ω as a functional of the vielbein e . This allows one to eliminate the fields $\tilde{\Psi}^{ab}$ and $\tilde{\Psi}^{ab}$.

Then, since the octonionic self-duality equation (1) is only $Spin(7) \subset SO(8)$ gauge invariant, one uses seven freedoms contained in the Lorentz antighost $\tilde{\Phi}$ to enforce that $\Omega^{ab-} = 0$.

As discussed before, the graviphoton Maxwell term $|F|^2$ is generated by the action in eq. (20). To eliminate the fermionic partner χ^+ of β^+ , one must then use seven other freedoms contained in $\tilde{\Phi}$. As evocated earlier, this imposes that one goes to a further decomposition under a G_2 subgroup of $Spin(7)$, which gives $\tilde{\Phi}^{ab} = \tilde{\Phi}_7^{ab-} + \tilde{\Phi}_7^{ab+} + \tilde{\Phi}_{14}^{ab+}$. We then add to our action the following s -exact term:

$$\int_{\mathcal{M}_8} s \left[\sqrt{g} \left(\tilde{\Phi}_7^{ab-} \Omega^{ab-} + \tilde{\Phi}_7^{ab+} (\chi^{ab+} - \Omega_7^{ab+}) \right) \right] \quad (21)$$

The gauge symmetry which arise for the fields $(\Psi^a, \bar{\Psi}, \bar{\Psi}^{ab-})$, is fixed by using the antighost $(\bar{\Phi}^a, \tilde{L}^{ab-}, \bar{\Phi})$ and their fermionic Lagrange multipliers exactly as in [1], with the s -exact action:

$$I_{ghosts} = \int d^8x s \left[\sqrt{g} (\bar{\Phi}^a D_\mu \Psi_\mu^a + \tilde{L}^{ab-} D_\mu \bar{\Psi}_\mu^{ab-} + \bar{\Phi} \partial_\mu \bar{\Psi}_\mu) \right] \quad (22)$$

This term, after untwisting, provides the gauge-fixing for the longitudinal parts of all components of the gravitino. This indicates that the fields $(\bar{\Phi}^a, \tilde{L}^{ab-}, \bar{\Phi})$ can be identified as the commuting Faddeev-Popov antighosts of $N = 1, D = 8$ local supersymmetry, while the fields $(\bar{\eta}^a, \tilde{\eta}^{ab-}, \eta)$ can be identified as the corresponding anti-commuting Nielsen-Kallosh ghosts.

After all, we see that the field $\tilde{\Phi}_{14}^{ab}$ in the topological multiplet of the spin connection eq. (8) is still not exploited. This is the signal that another topological multiplet can be introduced, whose gauge fixing conditions can be imposed using these remaining degrees of freedom. The mechanism here is analogous to the introduction of the graviphoton A in order to complete the topological gauge-fixing of the vielbein e and of the spin connection ω

[1] . In the present case the extra topological multiplet should be naturally identified with that of a two-form gauge field.

From the geometrical point of view, the presence of a two-form should induce a non trivial torsion, and the breaking to a G_2 holonomy group could be expected since this gives a natural way to define a torsion tensor $S^{ijk} \sim c^{ijk}$ proportional to the octonionic structure constants $c^{ijk} = \Omega^{ijk8}$, with $i, j, k = 1, \dots, 7$. We reserve to further work this construction.

5 Conclusion

In this paper we have shown that the determination of a gravitational topological field theory can be generalized from the case of four dimensions to that of eight dimensions. It is interesting to remark how the BRST transformations in our topological model generates local supersymmetry. In fact, this model appears related to a twisted version of $N = 1, D = 8$ supergravity. Although the situation is technically more involved than in four dimensions, the twist operation is of a conceptually simpler nature in eight dimensions. This is due to the existence of triality, which allows one to simply identify spinor and vector indices. The twist operation appears as a change of variables that can be defined on manifolds with holonomy group contained in $Spin(7)$. Such manifolds with G_2 and $Spin(7)$ holonomy have recently attracted a renewed attention in the context of M -theory compactifications [8]. An interesting result of our analysis is that on these manifolds the Einstein action can be written as a quadratic form in the anti-self-dual part of the spin connection, eq. (2). The relevant rôle, which is played in this context by the self-duality conditions on the spin connection, has been underlined also in [9, 10].

The observables of our topological model could be defined from the solution of descent equations starting from the eight form $\Omega R^{ab} R^{ab}$. It remains to identify such observables with known or unknown topological invariants of eight dimensional manifolds of special holonomy.

We find this result as a very encouraging step toward our goal which is to show that $N = 1, D = 8$ supergravity can be identified as an untwisted topological field theory. It must be noted that the dimensional reduction of our model in lower dimensions indicate that the low energy effective action of $N = 2, D = 4$ supergravity could be analysed à la Seiberg and Witten, in a way that parallels the treatment of the Yang–Mills theory of [4].

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